Roots of Polynomials, Integer Partitions, and L-Functions

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April 21 2023

Integer Partitions

Definition

An **integer partition of** n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that

$$\lambda_1 + \ldots + \lambda_k = n.$$

The number of partitions of n is denoted by $p(n)$.

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Example

The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Thus, $p(4) = 5$.

Jensen Polynomials

Definition

Let $a\colon \mathbb{N} \to \mathbb{R}$ be be an arithmetic function. The **Jensen polynomial of degree** d **and shift** n **associated to** a is

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Remark

With Taylor coefficients of an entire function f as our terms we obtain

$$J_f^{d,n}(z) = J_{f^{(n)}}^{d,0}(z) = \sum_{j=1}^{d} \binom{d}{j} f^{(n+j)}(0) z^j.$$

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Remark

The hyperbolicity of Jensen polynomials can encode information about a sequence.

Jensen Polynomials

Example

The roots of $J_a^{2,n}(z) = a_{n+2}z^2 + 2a_{n+1}z + a_n$ are

$$z = \frac{-a_{n+1} \pm \sqrt{(a_{n+1})^2 - a_{n+2}a_n}}{a_{n+2}}.$$

$J_a^{d,n}(z)$ is hyperbolic if and only if $(a_{n+1})^2 \geq a_{n+2}a_n$.

Jensen Polynomials Over $p(n)$

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A Look Ahead

Griffin, Ono, Rolen, and Zagier (2019) showed for every degree d, there exists an $N(d)$ such that if $n \geq N(d)$ then $J_p^{d,n}(z)$ is hyperbolic.

The Riemann Zeta and Xi Functions

Definition

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$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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The **Riemann Xi-function** is the entire function

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

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Remark

Riemann Hypothesis is true $\iff$ all zeros of Ξ are real.

Function Order

Definition

The **order** of a function f is given by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(f;r)}{\log r},$$

where $M(f;r)$ is the maximum modulus function.

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Theorem

The function Ξ has order 1.

The Laguerre-Pólya Class

Definition

A real entire function $\psi(z)$ belongs to the **Laguerre-Pólya class**, if

$$\psi(z) = Cz^m e^{bx-ax^2} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k}\right) e^{-\frac{z}{z_k}},$$

where $b, C, z_k \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $a \geq 0$ and $\sum_{k\geq 1} x_k^{-2} < \infty$.
If for $\psi(z) \in \mathcal{L} - \mathcal{P}$, either $\psi(z)$ or $\psi(-z)$ is

$$\psi(z) = Cz^m e^{\sigma x} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k}\right),$$

with $C \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $\sigma \geq 0$, $z_k > 0$, and $\sum_{k\geq 1} z_k^{-1} < \infty$ then we say ψ is **type I** and we denote $\psi \in \mathcal{L} - \mathcal{P}I$.

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Definition

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A sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ is a **multiplier sequence of type II** if $\Gamma_\gamma(p(x))$ has only real zeros whenever $p(x)$ has only real zeros with the same sign.

Multiplier Sequences, $\mathcal{L}-\mathcal{P}$, and Jensen Polynomials

Theorem (Pólya)

If $\{\gamma_k\}_{k\geq 0}$ is a sequence of nonnegative real numbers, then the following are equivalent:

1. $\{\gamma_k\}_{k\geq 0}$ *is a multiplier sequence.*

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2. *For each d, the polynomial $J_{\gamma}^{d,0}(z)$ has all real non-positive roots. Equivalently, $J_{\gamma}^{d,0}(z) \in \mathcal{L}-\mathcal{P}I$.*

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3. *The formal power series $\phi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z_k \in \mathcal{L}-\mathcal{P}I$.*

The Shifted Laguerre-Pólya Class

Definition

A real entire function $\phi(x)$ belongs to the **shifted Laguerre-Pólya class of degree** d, denoted $\mathcal{SL}-\mathcal{P}(d)$, if it's the uniform limit of polynomials $\{\phi_k\}_{k\geq 0}$ such that $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots for $n \geq N(d) \in \mathbb{N}$.

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We say $\phi \in \mathcal{SL}-\mathcal{P}(d)$ is of **type I** and write $\phi \in \mathcal{SL}-\mathcal{P}I(d)$ if all of the roots of $\phi_n^{(deg()\phi_n-d)}(x)$ have the same sign.

The Shifted Laguerre Pólya Class

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A real entire function $\phi_\gamma(x)$ belongs to the **shifted Laguerre-Pólya class**, denoted by $\mathcal{SL} - \mathcal{P}$, if $\phi_\gamma \in \mathcal{SL} - \mathcal{P}(d)$ for every $d \in \mathbb{N}$.

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If a real entire function $\phi \in \mathcal{SL}-\mathcal{P}$ satisfies

- $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots of the same sign for any $n \geq N(d)$ then it is **type I** and $\phi_\gamma \in \mathcal{SL}-\mathcal{P}I$.

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- $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots of the same sign for any $n \geq N(d)$ then it is **type I** and $\phi_\gamma \in \mathcal{SL}-\mathcal{P}I$.
- $\phi_\gamma \in \mathcal{SL}-\mathcal{P}I$ and $\gamma_k \geq 0$ for large enough k then we say $\phi_\gamma \in \mathcal{SL}-\mathcal{P}^+$

Properties of $\mathcal{SL} - \mathcal{P}$

1. $\mathcal{SL} - \mathcal{P}(d) \subset \mathcal{SL} - \mathcal{P}(d-1)$ for all $d \in \mathbb{N}$.

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   - If $\phi \in \mathcal{L}-\mathcal{P}$ and transcendental then $\phi \in \mathcal{SL}-\mathcal{P}(d)$ for any nonnegative integer d.
   - In this case we can take $N(d) = 0$ and consider $\mathcal{L}-\mathcal{P}$ as the shift 0 case of $\mathcal{SL}-\mathcal{P}$.

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   - In this case we can take $N(d) = 0$ and consider $\mathcal{L} - \mathcal{P}$ as the shift 0 case of $\mathcal{SL} - \mathcal{P}$.
3. Functions in $\mathcal{SL} - \mathcal{P}$ have order at most 2.
4. Functions in $\mathcal{SL} - \mathcal{P}I$ have order at most 1.

Shifted Multiplier Sequences

"Definition" (Wagner)

A real sequence $\{\gamma_k\}_{k\geq 0}$ is an **order d multiplier sequence of type I** if, for each $n \in \mathbb{N}$, $\{\gamma_k\}$ is a multiplier sequence when $deg(p) \geq d$.

A real sequence $\{\gamma_k\}_{k\geq 0}$ is a **shifted multiplier sequence of type I** (**type II** respectively) if for each $d \in N$, there exists an $N(d)$ such that $\{\gamma_{k+n}\}_{k\geq 0}$ is an order d multiplier sequence of type I (type II respectively) for all $n \geq N(d)$.

Shifted Analog of Pólya's Theorem

Theorem (Wagner)

If $\{\gamma_k\}_{k\geq 0}$ is a sequence of nonnegative real numbers, then the following are equivalent:

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First Motivating Results

Theorem (Griffin, Ono, Rolen, Zagier)

Let $a(n)$ be a real sequence with appropriate growth, then for each $d \geq 1$, all but (possibly) finitely many $J_a^{d,n}(X)$ are hyperbolic.

Appropriate Growth

Definition

A real sequence $a(n)$ has **appropriate growth** if for each j we have

$$a(n+j) = a(n)E(n)^j e^{-\delta(n)^2(j^2/4+o(1))},$$

as $n \to +\infty$ for some real numbers $E(n) > 0$ and $\delta(n) \to 0$.

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Remark

A sequence $a(n)$ with an asymptotic formula has appropriate growth if

$$\log\left(\frac{a(n+j)}{a(n)}\right) = A(n)j - B(n)j^2 + o(\delta(n)^2),$$

where $A(n) > 0$ and $0 < B(n) \to 0$.

Renormalized Jensen Polynomials

Definition

If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by,

$$\widehat{J}_a^{d,n}(X) := \frac{2^d}{\delta(n)^d \cdot a(n)} \cdot J_a^{d,n}\left(\frac{\delta(n)X - 1}{E(n)}\right).$$

Hermite Polynomials

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Classical Results

- Each $H_d(X)$ is hyperbolic with d distinct real roots.
- The Hermite polynomials have the exponential generating function

$$\sum_{d=0}^{\infty} H_d(X) \cdot \frac{Y^d}{d!} := e^{2XY - Y^2}.$$

Proving the Hyperbolicity of Jensen Polynomials

Theorem (Griffin, Ono, Rolen, Zagier)

Suppose $a(n)$ has appropriate growth. For each degree $d \geq 1$ we have,

$$\lim_{n \to +\infty} \widehat{J}_a^{d,n}(X) = H_d(X).$$

Thus, for each d, all but (possibly) finitely many $J_a^{d,n}(X)$ are hyperbolic.

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Thus, for each d, all but (possibly) finitely many $J_a^{d,n}(X)$ are hyperbolic.

Proof Idea

The general idea of the proof is to show that for large fixed n,

$$\sum_{d=0}^{\infty} \widehat{J}_a^{d,n}(X) \cdot \frac{Y^d}{d!} \approx e^{2XY - Y^2}.$$

Bounding the Hyperbolicity of Jensen Polynomials

Notation

Let $N(f; d)$ denote the minimal integer such that if $n \geq N(f, d)$ then $J_f^{d,n}(z)$ is hyperbolic.

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Main Theorem (Kim and Lee)

Let f be a transcendental real entire function of order $\rho < 2$ and $\mathcal{Z}(f) \subset \mathbb{S}$. Then, for every $c > \rho$ we have $N(f;d) = O(d^{c/2})$ as $d \to \infty$.

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Consequence

Then, $\rho(\Xi) = 1 \implies N(\Xi;d) = O(d^{1/2+\varepsilon})$ as $d \to \infty$!

Proof of Main Theorem of Kim and Lee

Notation

Let

- $\mathbb{S} := \{z \in \mathbb{C} : |\,\mathrm{Im}\,| \le \frac{1}{2}\}$
- $\mathcal{Z}(f)$ be the zero set of the function f
- $S(\delta) := \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| \le \delta |z|\}$

Proof of Main Theorem of Kim and Lee

Theorem (Kim)

Let f be a nonconstant real entire function with $0 < \rho(f) \leq 2$ and of minimal type. If there is a positive real number A such that $\mathcal{Z}(f) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \leq A\}$, then for any positive constant B there is a positive integer n_1 such that $f^{(n)}(z)$ has only real zeros in $|\operatorname{Re} z| \leq Bn^{\frac{1}{\rho}}$ for all $n \geq n_1$.

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Theorem 2 (Kim and Lee)

Let f be a transcendental real entire function of order $\rho(f) < 2$ and $\mathcal{Z}(f) \subset \mathbb{S}$. Then for every $c > \rho(f)$ there is a positive integer n_1 such that for all $n \geq n_1$

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S} : |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

Proof of Main Theorem of Kim and Lee

Theorem 3 of (Kim and Lee)

Let P and Q be real polynomials, $\delta > 0$, $Z(P) \subset S(\delta)$, Q is hyperbolic, and $deg(Q) \leq \delta^{-2}$. Then the polynomial

$$P(D)Q = \sum_{k=0}^{degP} \frac{P^{(k)}(0)}{k!} Q^{(k)},$$

is hyperbolic.

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Corollary (Kim and Lee)

Let P be a real polynomial with $\mathcal{Z}(P) \subset S(\delta)$ for $\delta > 0$. Then, $J_P^{d,0}(z)$ is hyperbolic for $d \leq \delta^{-2}$.

Proof of Main Theorem of Kim and Lee

- By Theorem 2, there exists an $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then

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- Let $d, n \in \mathbb{N}$ such that

$$n \geq \max\left\{ n_1, \left(\frac{d}{4}\right)^{c/2} \right\},$$

and choose $\delta = \frac{1}{2n^{1/c}} > 0$.

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and choose $\delta = \frac{1}{2n^{1/c}} > 0$.
Then, $d \leq 4n^{2/c} = (2n^{1/c})^2 = \left(\frac{1}{2n^{1/c}}\right)^{-2} = \delta^{-2}$.

Proof of Main Theorem of Kim and Lee

- Let $P_1, P_2, \ldots$ be real polynomials such that $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \cup \mathbb{R}$ for all k and $P_k \to f$ uniformly on compact subsets of $\mathbb{C}$ (these exist from partial products of the Weierstrass factorization of f).

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
- Since $d \leq \delta^{-2}$, $\mathbb{R} \subset S(\delta)$, and if $z \in \mathcal{Z}(f)$ with $|\operatorname{Re} z| \geq n^{1/c}$ then $|\operatorname{Im} z| \cdot 1 \leq \frac{1}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{1}{2n^{1/c}}|z| = \delta|z|$, the corollary applies.

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
- Since $d \leq \delta^{-2}$, $\mathbb{R} \subset S(\delta)$, and if $z \in \mathcal{Z}(f)$ with $|\operatorname{Re} z| \geq n^{1/c}$ then $|\operatorname{Im} z| \cdot 1 \leq \frac{1}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{1}{2n^{1/c}}|z| = \delta|z|$, the corollary applies.
- The corollary gives that $J_{P_k}^{d,0}(z)$ is hyperbolic for all k.

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
- Since $d \leq \delta^{-2}$, $\mathbb{R} \subset S(\delta)$, and if $z \in \mathcal{Z}(f)$ with $|\operatorname{Re} z| \geq n^{1/c}$ then $|\operatorname{Im} z| \cdot 1 \leq \frac{1}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{1}{2n^{1/c}}|z| = \delta|z|$, the corollary applies.
- The corollary gives that $J_{P_k}^{d,0}(z)$ is hyperbolic for all k.
- This implies $J_f^{d,n}$ is hyperbolic with

$$N(f;d) \leq \left\lceil \max\left\{n_1, \left(\frac{d}{4}\right)^{c/2}\right\}\right\rceil,$$

or, equivalently, $N(f;d) = O(d^{c/2})$ as $d \to \infty$. ∎

Dirichlet Characters

Definition

A **Dirichlet character modulo** k is a function $\chi\colon \mathbb{N} \to \mathbb{C}$ satisfying

(i) $\chi(1) = 1$;

(ii) $\chi(n_1) = \chi(n_2)$ if $n_1 \equiv n_2 \pmod{k}$;

(iii) $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$;

(iv) $\chi(n) = 0$ if and only if $(n, k) > 1$.

Dirichlet Characters

Example

There are four Dirichlet characters modulo 5, namely

$n \pmod 5$	1	2	3	4	0
$\chi_0(n)$	1	1	1	1	0
$\chi_1(n)$	1	i	$-i$	-1	0
$\chi_2(n)$	1	-1	-1	1	0
$\chi_3(n)$	1	$-i$	i	-1	0

The Dirichlet character χ_0 is called the **principal character**.

Dirichlet L-Functions

Definition

Let χ be any character modulo k. The **Dirichlet series for** χ is

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s},$$

for any real $s > 1$.

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Remark

Dirichlet L-functions act similarly to ζ, including having completed form and analytic continuation to $\mathbb{C}$ as well as nontrivial zeros contained in the strip $0 < \operatorname{Im} z < 1$.

Can we generalize the methods of Kim and Lee?

Definition (Wagner)

For a Dirichlet L-function $L(\chi, s)$, let $\Lambda(\chi, s)$ denote its completed form. We formally define

$$\Xi(\chi, z) := \begin{cases} \left(-z^2 - \frac{1}{4}\right) \Lambda\left(\frac{1}{2} - iz, \chi\right) & \text{if } \chi \text{ is principal} \\ \Lambda\left(\frac{1}{2} - iz, \chi\right) & \text{otherwise} \end{cases}.$$

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Remark

We can note that $\Xi(\chi, z)$ is transcendental real entire, with $\mathcal{Z}(\Xi(\chi, z)) \subset \mathbb{S}$.
Thus, we can apply the Main Theorem of Kim and Lee if we verify $\rho(\Xi(\chi, z)) < 2$.

Order of $\Xi(\chi, z)$

Theorem

Let $L(s, \chi)$ be a Dirichlet L-function. Then, $\rho(\Xi(\chi, z)) = 1$.

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We present a proof of this fact which does not exist in the literature to the author's knowledge.

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We present a proof of this fact which does not exist in the literature to the author's knowledge.

Additionally, we will use the following implications

$$\rho(\Xi(\chi, z)) = 1 \iff \rho(L(\chi, s)) = 1 \iff \rho((s-1) \cdot L(\chi, s)) = 1.$$

Proof that $\Xi(\chi, z)$ has order 1

- We first consider the Laurent series for $L(s,\chi)$ at $s=1$,

$$L(s,\chi) = \frac{\delta_\chi}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^n,$$

and multiplying by $s-1$ yields

$$(s-1)L(s,\chi) = \delta_\chi + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^{n+1}.$$

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- We can express the order in terms of the coefficients of the Laurent series as

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{\log\left(\left|\frac{(-1)^n\gamma_n(\chi)}{n!}\right|^{-1}\right)} = \limsup_{n\to\infty} \frac{n \log n}{-\log\left(\frac{|\gamma_n(\chi)|}{n!}\right)}.$$

Proof that $\Xi(\chi, z)$ has order 1

Theorem (Saad Eddin)

Let χ be a primitive Dirichlet character modulo q. Then, for every $1 \le q \le \frac{\pi}{2} \cdot \frac{e^{(n+1)/2}}{n+1}$, we have

$$\frac{|\gamma_n(\chi)|}{n!} \le q^{-\frac{1}{2}} C(n,q) \min\left(1 + D(n,q), \frac{\pi^2}{6}\right),$$

where

$$C(n,q) \sim \exp\left\{-n \log\theta(n,q) + \theta(n,q)\log\theta(n,q) + \theta(n,q)O(1)\right\},$$

$$\theta(n,q) \sim \frac{n}{\log n}, \qquad D(n,q) = 2^{-\theta(n,q)-1}.$$

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- First,

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$$\log C(n,q) \sim \log\exp\{-n\log\theta(n,q) + \theta(n,q)\log\theta(n,q) + \theta(n,q)\}$$
$$\sim -n\log n + n\log\log n + \frac{n}{\log n}(\log n - \log\log n) + \frac{n}{\log n}$$
$$= -n\log n + O(n\log\log n).$$

Proof that $\Xi(\chi, z)$ has order 1

- We return to our equation for the order of $(s-1)L(\chi, s)$

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{-\log\left(\frac{|\gamma_n(\chi)|}{n!}\right)}$$

$$\leq \limsup_{n\to\infty} \frac{n \log n}{-\log\left(q^{-\frac{1}{2}} C(n,q) \min\left(1 + D(n,q), \frac{\pi^2}{6}\right)\right)}$$

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Proof that $\Xi(\chi, z)$ has order 1

- By contradiction assume $\rho(L(\chi, s)) < 1$.
- Then $L(\chi, s)$ has genus 0 by Hadamard's Theorem.
- This implies the zeros of ζ are "sparse."
- We have the following recent result giving a bound on the number of zeros of Dirichlet L-functions.

Theorem (Bennett, Martin, O'Bryant, Rechnitzer)

Suppose that the Dirichlet character χ has conductor $q > 1$, and that $T \geq 5/7$. Then, the number of zeros of $L(\chi, s)$ and height at most T, $N(T, \chi)$, is bounded by

$$\left| N(T, \chi) - \left(\frac{T}{\pi} \log \frac{qT}{2\pi e} - \frac{\chi(-1)}{4} \right) \right| \leq 0.22737\ell + 2\log(1+\ell) - 0.5,$$

where $\ell = \log \frac{q(T+2)}{2\pi} > 1.567$.

Proof that $\Xi(\chi, z)$ has order 1

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- The number of zeros satisfies $N(T, \chi) \sim T \log T$.
- This is too many zeros for a genus 0 function so we reach a contradiction.
- Since $1 \leq \rho(\Xi(\chi, z)) \leq 1$, we have that $\rho(\Xi(\chi, z)) = 1$. ■

Generalizing to Dirichlet L-functions

Theorem

Let χ be a principal character modulo q, let $L(\chi, s)$ be a Dirichlet L-function, and let $\Xi(\chi, z)$ be defined as above. Then, $N(\Xi(\chi, z); d) = O(d^{\frac{1}{2}+\varepsilon})$ as $d \to \infty$.

Proof.

The function $\Xi(\chi, z)$ is a transcendental real entire function with order $\rho = 1 < 2$ and $\mathcal{Z}(\Xi(\chi, z)) \subset \mathbb{S}$. Choose $c = 1 + \varepsilon_0 > \rho$ for arbitrarily small $\varepsilon_0 > 0$. Then, by the Main Theorem of Kim and Lee, we have that $N(\Xi(\chi, z); d) = O(d^{(1+\varepsilon_0)/2}) = O(d^{\frac{1}{2}+\varepsilon})$ as $d \to \infty$. ∎

L-functions

Definition

A **Dirichlet series** is a series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $s \in \mathbb{C}$ and $\{a_n\}_{n \geq 1}$ is a sequence of complex numbers.

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Definition

If a Dirichlet series $L(s)$ admits an meromorphic continuation, it is called an L-**series**, and its continuation is called an L-**function**.

Good L-Functions

Definition

A Dirichlet series $L(s)$ is **good** if the following hold

1. $L(s)$ has a completed form, $\Lambda(s)$ which has an integral representation.

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3. The coefficients of $\Lambda(s)$ are real.
4. $\rho(\Lambda(s)) < 2$.

Generalizing to L-functions

Definition

For a good Dirichlet $L(s)$ series with completed form $\Lambda(s)$, we define

$$\Xi_L(z) := \begin{cases} (-z^2 - \frac{k^2}{4})\Lambda(\frac{k}{2} - iz) & \text{if } \Lambda(s) \text{ has a pole at } s = k \\ \Lambda(\frac{k}{2} - iz) & \text{otherwise.} \end{cases}$$

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Remarks

- The function $\Xi_L(z)$ is transcendental, real, and entire.

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Remarks

- The function $\Xi_L(z)$ is transcendental, real, and entire.
- We have $\rho(\Xi_L) < 2$ by definition.
- The zero set satisfies $\mathcal{Z}(\Xi_L) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \leq k/2\} := \mathbb{S}_k$.

Generalizing to L-Functions

Theorem

Let $L(s)$ be a good Dirichlet series. Then, $N(\Xi_L; d) = O(d)$ for $\varepsilon > 0$ as $d \to \infty$.

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Proof

- We modify the methods of Kim and Lee.

Generalizing to L-Functions

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Let $L(s)$ be a good Dirichlet series. Then, $N(\Xi_L; d) = O(d)$ for $\varepsilon > 0$ as $d \to \infty$.

Proof

- We modify the methods of Kim and Lee.
- We want an analog to their second theorem, which is only stated for functions satisfying $\mathcal{Z}(f) \subset \mathbb{S}$.

Proof of the L-Function Bound Continued

Theorem (Kim)

Let f be a nonconstant real entire function with $0 < \rho(f) \leq 2$ and of minimal type. If there is a positive real number A such that $\mathcal{Z}(f) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \leq A\}$, then for any positive constant B there is a positive integer n_1 such that $f^{(n)}(z)$ has only real zeros in $|\operatorname{Re} z| \leq Bn^{\frac{1}{\rho}}$ for all $n \geq n_1$.

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Theorem 2 (Kim and Lee)

Let f be a transcendental real entire function of order $\rho(f) < 2$ and $\mathcal{Z}(f) \subset \mathbb{S}$. Then for every $c > \rho(f)$ there is a positive integer n_1 such that for all $n \geq n_1$

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S} : |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

Proof of the L-Function Bound Continued

Theorem

Let f be a transcendental real entire function of order $\rho(f) < 2$ and $\mathcal{Z}(f) \subset \mathbb{S}_k$ for some $k \in \mathbb{R}$. Then for every $c > \rho(f)$ there is a positive integer n_1 such that for all $n \geq n_1$

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S}_k \,:\, |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

Proof.

We apply the theorem of Kim, choosing $A = k/2$ (rather than $1/2$) and choose $B = 1$. Then, taking into account the lack of minimal type condition, for any $c > \rho$ Kim's theorem implies there exists an $n_1 \in \mathbb{N}$ such that $f^{(n)}(z)$ has only real zeros in $|\operatorname{Re} z| \leq n^{1/c}$ when $n \geq n_1$. This implies the theorem. ∎

Proof of the L-Function Bound Continued

- By the previous theorem, there exists an $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then

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- Let $d, n \in \mathbb{N}$ such that

$$n \geq \max\left\{n_1, \left(\frac{k^2}{4} \cdot d\right)^{c/2}\right\},$$

and choose $\delta = \frac{k}{2n^{1/c}} > 0$.

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and choose $\delta = \frac{k}{2n^{1/c}} > 0$.
Then, $d \leq \frac{4}{k^2} n^{2/c} = (\frac{2}{k} n^{1/c})^2 = \left(\frac{k}{2n^{1/c}}\right)^{-2} = \delta^{-2}$.

Proof of the L-Function Bound Continued

- Let $P_1, P_2, \ldots$ be real polynomials such that $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \cup \mathbb{R}$ for all k and $P_k \to f$ uniformly on compact subsets of $\mathbb{C}$ (these exist from partial products of the Weierstrass factorization of f).

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
- Since $d \leq \delta^{-2}$, $\mathbb{R} \subset S(\delta)$, and if $z \in \mathcal{Z}(f)$ with $|\operatorname{Re} z| \geq n^{1/c}$ then $|\operatorname{Im} z| \cdot 1 \leq \frac{k}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{k}{2n^{1/c}}|z| = \delta|z|$, the corollary of Kim and Lee applies.

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- Let $P_1, P_2, \ldots$ be real polynomials such that $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \cup \mathbb{R}$ for all k and $P_k \to f$ uniformly on compact subsets of $\mathbb{C}$ (these exist from partial products of the Weierstrass factorization of f).
- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
- Since $d \leq \delta^{-2}$, $\mathbb{R} \subset S(\delta)$, and if $z \in \mathcal{Z}(f)$ with $|\operatorname{Re} z| \geq n^{1/c}$ then $|\operatorname{Im} z| \cdot 1 \leq \frac{k}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{k}{2n^{1/c}}|z| = \delta|z|$, the corollary of Kim and Lee applies.
- The corollary gives that $J_{P_k}^{d,0}(z)$ is hyperbolic for all k.

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- This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$.
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- The corollary gives that $J_{P_k}^{d,0}(z)$ is hyperbolic for all k.
- This implies $J_f^{d,n}(z)$ is hyperbolic with

$$N(f;d) \leq \left\lceil \max\left\{ n_1, \left(\frac{k^2}{4} \cdot d\right)^{c/2} \right\} \right\rceil,$$

or equivalently $N(f;d) = O(d^{c/2})$ as $d \to \infty$.

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Proof of the L-Function Bound Continued

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or equivalently $N(\Xi_L; d) = O(d)$ as $d \to \infty$. ■

Why Do We Care?

Background

- Pólya showed that RH is equivalent to the hyperbolicity of all of the $J_{\gamma}^{d,n}(z)$ where γ are Taylor coefficients of Ξ.

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Remark

The bound on $N(\Xi_L; d)$ provides further evidence for GRH.

Can We Generalize to $\mathcal{SL} - \mathcal{P}$?

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- For any good L-Function, $\Xi_L(z) \in \mathcal{SL} - \mathcal{P}$ (Wagner).
- Generalization of the methods of Kim and Lee to all of $\mathcal{SL} - \mathcal{P}$ (or at least $\mathcal{SL} - \mathcal{P}I$) seems natural.

Problems With Generalizing to $\mathcal{SL} - \mathcal{P}$

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- We can't use partial products of Weierstrass factorization as sequence of polynomials converging to ϕ as they are not contained in $S(\delta)$ for any $\delta > 0$.

How Would The Generalization Work?

Philosophy of $\mathcal{SL} - \mathcal{P}$

For $\phi_\gamma \in \mathcal{SL} - \mathcal{P}$, the Taylor coefficients γ_k should act more and more like Taylor coefficients of a function in $\mathcal{L} - \mathcal{P}$ as k grows.

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- There should exist an analog of Theorem 2 of Kim and Lee for functions in $\mathcal{L} - \mathcal{P}$.
- There may exist some $n_\delta \in \mathbb{N}$ such that if $n \geq n_\delta$ then $\phi(z)$ hyperbolic for $|\operatorname{Re} z| \leq n^{1/c}$.

Results

Theorem (Wagner)

For $\phi_\gamma \in \mathcal{SL} - \mathcal{P}$, the sequence of polynomials

$$P_{n,k}(z) := J_\gamma^{k,n}(z/k) = \sum_{j=0}^{k} \binom{k}{j} \gamma_{n+j} (z/k)^j$$

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Theorem

If z_0 is a root of $P_{k,n}(z)$ then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|z_0| \leq k^2 \cdot \frac{\gamma_{n+k-1}}{\gamma_{n+k}}.$$

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When $k \leq d$, we keep the same choice of $P_{k,n} = J_{\gamma}^{k,n}(z/k)$ but for $k > d$ we define

$$\widehat{P}_{d,k,n}(z) := \sum_{j=0}^{d} \binom{k}{j} \gamma_{n+j} (z/k)^j.$$

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For $k \leq d$, $P_{k,n}(z) = \widehat{P}_{d,k,n}(z)$ and when $k > d$ they have the same first d Taylor coefficients, so $J_P^{d,0}(z) = J_{\widehat{P}}^{d,0}(z)$ for all k.

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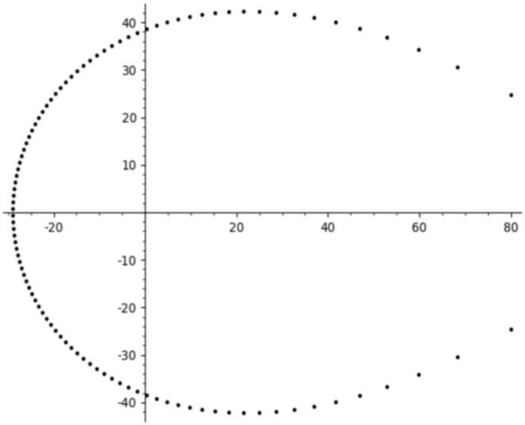
Question

Do $\widehat{P}$ satisfy $\mathcal{Z}(\widehat{P}) \subset S(\delta)$ for some $\delta > 0$ and large k and n?

Roots of $\widehat{P}_{d,k,n}(z)$ with $d = 100$, $k = 10^9$, $n = 10^9$



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If z_0 is a root of $\widehat{P}_{d,k,n}(p(i);z)$ and $n \geq 26$ then $|z_0| \leq d^2$.

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Proof.

By the previous general theorem and using that $p(i)$ is increasing

$$|z_0| \leq \begin{cases} k^2 \cdot \frac{p(n+k-1)}{p(n+k)} & k \leq d \\ d \cdot \frac{k}{k-d+1} \cdot \frac{p(n+d-1)}{p(n+d)} & k > d \end{cases}$$

$$\leq \begin{cases} d^2 \cdot 1 & k \leq d \\ d \cdot d \cdot 1 & k > d \end{cases} = d^2 \ \forall k.$$

∎

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$$n \geq \max\{n_\delta, 26, d^{5c/2}\}$$

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- Then $d \leq \delta(d)^{-2}$ and $\mathcal{Z}(\widehat{P}) \subset S(\delta(d))$ so Corollary of Kim and Lee would apply.
- Then $N(\phi_p; d) \leq \lceil \max\{n_\delta, 26, d^{5c/2}\} \rceil$ which would imply $N(\phi_p; d) = O(d^{5\cdot 1+\varepsilon/2}) = O(d^{5/2+\varepsilon})$ as $d \to \infty$.

Conclusion

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- For Dirichlet L-Functions, $N(\Xi(\chi, z); d) = O(d^{1/2+\varepsilon})$ as $d \to \infty$.

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- Proving that $N(\phi_p; d) = O(d^{5/2+\varepsilon})$.
- Final goal of generalizing the bound on $N(\phi; d)$ to all of $\mathcal{SL} - \mathcal{P}I$ and potentially all of $\mathcal{SL} - \mathcal{P}$.

Thank You!