The Dedekind Eta Function and Half-Integral Weight Modular Forms

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Integer Partitions

Definition

An **integer partition of** n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ such that

$$\lambda_1 + ... + \lambda_k = n.$$

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Example

The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Thus, $p(4) = 5$.

Partition Generating Function

Proposition

The generating function for $p(n)$ has the following product expression

$$P(q) := \sum_{k \geq 0} p(k) q^k = \prod_{n \geq 1} \frac{1}{1 - q^n}.$$

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- $\sum_{n \geq 0} p(n) q^n = (1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots)\ldots$.
- $\sum_{n \geq 0} p(n) q^n = \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\ldots = \prod_{n \geq 1} \frac{1}{1-q^n}$

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Remark

We proved in class that $\Delta \in S_{12}$ *and further* $S_{12} = \langle \Delta \rangle$

Another Property of Δ

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Can we learn anything about $p(n)$ from $\Delta(\tau)$ and the theory of modular forms? Yes!

Defining The Dedekind Eta Function

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Remark

The function $\eta(\tau)$ *satisfies*

$$(2\pi)^{12}\eta(\tau)^{24} = \Delta(\tau), \qquad P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}.$$

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Question

- *Does η transform "nicely" over $SL_2(\mathbb{Z})$?*
- *What does it mean to be a modular form with half-integral weight?*

A Quick Detour

Our initial guess for a transformation of a modular form with weight $k/2$, k odd, over $\Gamma = SL_2(\mathbb{Z})$ might be of the form

$$f(\gamma\tau) = (c\tau + d)^{k/2} f(\tau),$$

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This does NOT work!

Automorphic Factors

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Is $j(\gamma,\tau) := (c\tau + d)^{k/2}$ an automorphic factor?

Counterexample

Consider

$$\gamma_1 = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

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We can fix this though, since it almost works.

Modular Forms With Half-Integral Weight

Define

$$j(\gamma;\tau) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (c\tau + d)^{1/2},$$

where $\left(\frac{c}{d}\right)$ is a Kronecker character and

$$\varepsilon_d^{-1} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

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Definition

A modular form of weight $k/2$ with k odd on $\Gamma_0(4N)$ is a function $f : \mathbb{H} \to \mathbb{C}$ that is is holomorphic on $\mathbb{H}$ and at infinity and such that for all $\gamma \in \Gamma_0(4N)$ we have

$$f(\gamma\tau) = j(\gamma;\tau)^k f(\tau).$$

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Consider how η transforms under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have that

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{i\pi(\tau+1)}{12}} \prod_{n\geq 1} \left(1 - e^{2\pi i(\tau+1)n}\right) \\ &= e^{\frac{i\pi}{12}} q^{\frac{1}{24}} \prod_{n\geq 1} \left(1 - q^n e^{2\pi i n}\right) \\ &= e^{\frac{i\pi}{12}} \eta(\tau). \end{aligned}$$

Transformation of $\eta(\tau)$ on S

The transformation of η on $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is given by

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This can be proved using properties of the logarithmic derivative of $\eta(\tau)$ or using the Euler pentagonal number theorem and the representation of η as a theta series

$$\eta(\tau) = \sum_{n\geq 1} \chi_{12}(n) q^{\frac{n^2-1}{24}}.$$

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These can be expanded into a full multiplier system for $\eta(\tau)$.

Using the Modularity of $\eta(\tau)$

The modularity of $\eta(\tau)$ has been the key for many famous results about partitions, including the Hardy-Ramanujan asymptotic.

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Theorem (Hardy-Ramanujan)

The partition function $p(n)$ satisfies

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

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Hardy and Ramanujan invented the "circle method" to prove this, which computes the coefficients of $P(q)$ near the singularities of $1/\eta(\tau)$ at roots of unity.

Thank You!

References

- Kong, Y., Teo, L. *An Elementary Proof of the Transformation Formula for the Dedekind Eta Function.* arxiv: 2302.03280v1
- Rolen, L. *Modular Forms: Theory and Applications*. Lecture notes on https://math.vanderbilt.edu/rolenl/