Some Partition-Theoretic Analogies in Analytic Number Theory

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Philosophy of the Talk

- A rising topic of interest in partition theory over the last few years is in partition-theoretic analogs of functions, theorems, etc. from classical number theory

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- A rising topic of interest in partition theory over the last few years is in partition-theoretic analogs of functions, theorems, etc. from classical number theory
- The purpose of this talk is to introduce some new partition-theoretic analogs which generalize ideas from analytic number theory
- Our philosophy is that unifying partition theory and "classical" number theory is mutually beneficial

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Integer Partitions

Definition

An **integer partition of** n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that

$$\lambda_1 + \ldots + \lambda_k = n.$$

The number of partitions of n is denoted by $p(n)$.

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Example

The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Thus, $p(4) = 5$.

Partition Definitions

Definition

Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a partition. Then:

- each λ_i is a **part** within the partition λ
- we call $\ell(\lambda) = r$ the **length** of the partition λ
- we call $|\lambda| = \sum \lambda_i$ the **size** of λ (or number being partitioned)

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Notation

- we write "$\lambda \vdash n$" to say λ is a partition of n
- we write "$\lambda_i \in \lambda$" to denote that λ_i is one of the parts of λ
- we denote the set of all partitions by $\mathcal{P}$
- we denote the set of partitions with parts from $\mathbb{X} \subset \mathbb{Z}^+$ by $\mathcal{P}_{\mathbb{X}}$

An Interesting Bijection

- There exists a natural bijection between $\mathcal{P}_{\mathbb{P}}$ and $\mathbb{Z}^+$

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Under this bijection, we associate

$$7 \longleftrightarrow (7) \qquad 12 \longleftrightarrow (3,2,2) \qquad 36 \longleftrightarrow (3,3,2,2)$$

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Remark

This is completely multiplicative; this map says nothing about the additive nature of the associated partitions.

The Partition Norm

Definition

We define the **norm** of a partition $\lambda \in \mathcal{P}$, denoted n_λ, to be the product of its parts if $\lambda \neq \emptyset$ or $n_\emptyset := 1$ of $\lambda = \emptyset$.

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Question

Can we introduce operations on partitions that agree with multiplication/division of integers under the norm map?

A Multiplicative Theory of Partitions

Definition

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Example

Let $\lambda = (5,2,2)$ and $\delta = (6,5,1)$. Then

$$\lambda\delta = (5,2,2)(6,5,1) = (6,5,5,2,2,1).$$

A First Partition-Theoretic Analog

Definition

We define the **partition-theoretic Möbius function** for some $\lambda \in \mathcal{P}$ by

$$\mu_{\mathcal{P}}(\lambda) = \begin{cases} 0 & \text{if } \lambda \text{ has repeated parts,} \\ (-1)^{\ell(\lambda)} & \text{otherwise.} \end{cases}$$

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Compare to $\mu(n) = \begin{cases} 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ primes} \end{cases}.$

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Fundamental Rule For Partition-Theoretic Analogs

In general, a partition-theoretic (PT) analog of a function must equal its classical counterpart evaluated on the partition norm when restricted to prime partitions.

Analogous Properties For $\mu_{\mathcal{P}}$

Möbius Function $\mu(n)$	**PT Möbius Function $\mu_{\mathcal{P}}(\lambda)$**
Natural number n	Partition λ
Prime factors of n	Parts of λ
Nonzero for squarefree integers	Nonzero for partitions into distinct parts
$\sum_{d\mid n} \mu(d) = 0$	$\sum_{\delta\mid\lambda} \mu_{\mathcal{P}}(\delta) = 0$

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Proposition (PT Möbius Inversion, Schneider, 2018)

Let $f : \mathcal{P} \to \mathbb{C}$ *and define* $F(\lambda) := \sum_{\delta\mid\lambda} f(\delta)$. *Then*

$$f(\lambda) = \sum_{\delta\mid\lambda} F(\delta)\mu_{\mathcal{P}}(\lambda/\delta).$$

Partition-Theoretic Zeta Functions

Definition

Let $\mathcal{P}' \subseteq \mathcal{P}$. Then for $\mathrm{Re}(s) > 1$, we define the **partition-theoretic zeta function** over $\mathcal{P}'$ by

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} \frac{1}{n_\lambda^s}.$$

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- $\zeta_{\mathcal{P}_{\mathbb{P}}}(s) = \zeta(s)$ the classical Riemann zeta function
- if $1 \in \mathbb{X}$ then $\zeta_{\mathcal{P}_{\mathbb{X}}}(s)$ diverges for all $s \in \mathbb{C}$
- we have an Euler product of the form $\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} \frac{1}{1 - n^{-s}}$

The Prime Number Theorem

Definition

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Theorem (Prime Number Theorem)

The prime counting function has asymptotic

$$\pi(x) \sim \frac{x}{\log x}.$$

The von Mangoldt and Chebyshev Functions

Definition

We define the **von Mangoldt function** by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

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$$\vartheta(x) = \sum_{p \leq x} \log p.$$

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We define the **Chebyshev ψ function** by

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Equivalence of Asymptotics

Theorem (Chebyshev, 1852)

The following are equivalent:

1. $\pi(x) \sim \dfrac{x}{\log x}$;
2. $\vartheta(x) \sim x$;
3. $\psi(x) \sim x$.

An Exact Formula for ψ

Theorem (von Mangoldt, 1895)

The function $\psi_0(x) := \sum'_{n \leq x} \Lambda(n)$ *has exact form*

$$\psi_0(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ \text{nontrivial}}} \frac{x^\rho}{\rho} - \log(1 - x^{-2}) - \log(2\pi).$$

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Remark

PNT $\iff \psi(x) \sim x \iff \zeta(s)$ has no zeros on $\mathrm{Re}(s) = 1$.

Exact Formula Proof Outline

Proof Outline

Using Perron's Formula and the Residue Theorem

$$\begin{aligned}\psi_0(x) = {\sum_{n\le x}}' \Lambda(n) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)\cdot\frac{x^s}{s}ds \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\cdot\frac{x^s}{s}ds \\ &= -\sum \mathrm{Res}\left[\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{x^s}{s}\right] \\ &= x - \sum_{\substack{\zeta(\rho)=0 \\ \text{nontrivial}}}\frac{x^\rho}{\rho} - \log(1-x^{-2}) - \log(2\pi).\end{aligned}$$

Questions

The Little Question

How do we define partition-theoretic analogs of these functions that satisfy similar properties?

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The Big Question

Can we use these new PT analogs to prove analogs of classical results or even completely new results broadly in number theory?

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The Partition von Mangoldt Function

Definition

For $\lambda \in \mathcal{P}$ we define the **partition-theoretic von Mangoldt function** $\Lambda_{\mathcal{P}}(\lambda) \colon \mathcal{P} \to \mathbb{R}$ by

$$\Lambda_{\mathcal{P}}(\lambda) := \begin{cases} \log(n) & \text{if } \lambda = (n)^k = (n, \ldots, n) \text{ of length } k, \\ 0 & \text{otherwise.} \end{cases}$$

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Remark

For $\lambda \in \mathcal{P}_{\mathbb{P}}$ we have $\Lambda_{\mathcal{P}}(\lambda) = \Lambda(n_\lambda)$

Analogs of Theorems for $\Lambda(n)$

Classical	**PT Analog**
prime number p	$n \in \mathbb{X}$, for $\mathbb{X} \subseteq \mathbb{Z}^+$
prime factors of $n \in \mathbb{Z}^+$	parts of $\lambda \in \mathcal{P}$
d dividing n $(d \mid n)$	δ subpartition of λ $(\delta \mid \lambda)$
$\sum_{d \mid n} \Lambda(d) = \log(n)$	$\sum_{\delta \mid \lambda} \Lambda_{\mathcal{P}}(\delta) = \log(n_\lambda)$
$\Lambda(n) = -\sum_{d \mid n} \mu(d) \log(d)$	$\Lambda_{\mathcal{P}}(\lambda) = -\sum_{\delta \mid \lambda} \mu_{\mathcal{P}}(\delta) \log(n_\delta)$

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$\Lambda(n) = -\sum_{d\|n} \mu(d) \log(d)$	$\Lambda_{\mathcal{P}}(\lambda) = -\sum_{\delta\|\lambda} \mu_{\mathcal{P}}(\delta) \log(n_\delta)$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \quad \longleftrightarrow \quad \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \frac{\Lambda_{\mathcal{P}}(\lambda)}{n_\lambda^s} = -\frac{\zeta'_{\mathcal{P}_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}.$$

The Generalized Chebyshev Functions

Definition

For $\mathbb{X} \subseteq \mathbb{Z}^+$ and $x \in \mathbb{R}$, we define the **generalized Chebyshev** ϑ **function** by

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$$\psi_{\mathcal{P}'}(x) := \sum_{\substack{\lambda \in \mathcal{P}' \\ n_\lambda \leq x}} \Lambda_{\mathcal{P}}(\lambda).$$

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Compare to

$$\vartheta(x) = \sum_{\substack{p \in \mathbb{P} \\ p \leq x}} \log p, \qquad \psi(x) = \sum_{n \leq x} \Lambda(n).$$

Exact Formulas for PT Chebyshev Functions

Question

Can we find an exact formula for $\psi_{\mathcal{P}'}(x)$ *for some subsets* $\mathcal{P}' \subseteq \mathcal{P}$*?*

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First we need some tools from analysis...

A Partition-Theoretic Analog of Partial Summation

Proposition (Partition-Theoretic Partial Summation)

Let $a\colon \mathcal{P}' \to \mathbb{C}$ be a partition-theoretic function on $\mathcal{P}' \subseteq \mathcal{P}$ and let f be a continuously differentiable function on $[x, y]$. Then,

$$\sum_{\substack{\lambda \in \mathcal{P}' \\ x < n_\lambda \leq y}} a(\lambda) f(n_\lambda) = \mathcal{A}(y) f(y) - \mathcal{A}(x) f(x) - \int_x^y \mathcal{A}(t) f'(t) dt,$$

where

$$\mathcal{A}(t) := \sum_{\substack{\lambda \in \mathcal{P}' \\ n_\lambda \leq t}} a(\lambda).$$

A Partition-Theoretic Analog of Perron's Formula

Proposition (Partition-Theoretic Perron's Formula)

Let $a\colon \mathcal{P}' \to \mathbb{C}$ *be a partition-theoretic function on* $\mathcal{P}' \subseteq \mathcal{P}$ *with partition-theoretic Dirichlet series* $g(s) := \sum_{\lambda\in\mathcal{P}'} a(\lambda) n_\lambda^{-s}$ *such that* $g(s)$ *is uniformly convergent for* $\mathrm{Re}(s) \geq \sigma$. *Then, for any* $c > \sigma$

$$\widetilde{\mathcal{A}}(x) := \sum_{\substack{\lambda\in\mathcal{P}' \\ n_\lambda \leq x}}{}' a(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\frac{x^s}{s}\,ds.$$

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Remark

The primed sum indicates that the last term must be multiplied by $\frac{1}{2}$ *when* x *is an integer.*

Integral Representations for PT ψ Functions

Proposition

Let $\mathbb{X} \subseteq \mathbb{Z}^{+} \backslash \{1\}$*. If* $\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \Lambda_{\mathcal{P}}(\lambda) n_{\lambda}^{-s}$ *converges for* $\mathrm{Re}(s) > \sigma$*, then for* $c > \sigma$ *we have*

$$\widetilde{\psi}_{\mathbb{X}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'_{\mathcal{P}_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)} \right) \frac{x^s}{s} ds.$$

Proof of the Proposition

Proof.

$$\widetilde{\psi}_{\mathbb{X}}(x) = \sum_{\substack{\lambda \in \mathcal{P}_{\mathbb{X}} \\ n_\lambda \leq x}} \Lambda_{\mathcal{P}}(\lambda)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \frac{\Lambda_{\mathcal{P}}(\lambda)}{n_\lambda^s} \right) \frac{x^s}{s} ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'_{\mathcal{P}_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)} \right) \frac{x^s}{s} ds$$

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Remark

Evaluating the integral requires analytic continuation of $\zeta_{\mathcal{P}_{\mathbb{X}}}(s)$.

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A classical result says

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and it is widely conjectured that

$$Q(x) = \frac{6}{\pi^2}x + O\left(x^{\frac{1}{4}+\varepsilon}\right).$$

Best Known Error Bounds For $Q(x)$

Theorem (Liu, 2015)

Assuming the Riemann Hypothesis, we have

$$Q(x) = \frac{6}{\pi^2}x + O\left(x^{\frac{11}{35}+\varepsilon}\right).$$

Best Known Error Bounds For $Q(x)$

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Theorem (Walfisz, 1963)

The squarefree counting function satisfies

$$Q(x) = \frac{6}{\pi^2}x + O\left(x^{\frac{1}{2}} \exp\left\{-c\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right\}\right),$$

for some positive constant c.

"PNT" Equivalence For Squarefree Numbers

Theorem

Let $c > \frac{1}{4}$ and $\mathbb{S} = \{n \geq 2 \colon n \text{ squarefree}\}$. The following are equivalent:

1. $\pi_{\mathbb{S}}(x) = \frac{6}{\pi^2}x + O(x^c)$,
2. $\vartheta_{\mathbb{S}}(x) = \frac{6}{\pi^2}(x \log x - x) + O\big(x^c \log x\big)$,
3. $\psi_{\mathbb{S}}(x) = \frac{6}{\pi^2} \sum_{k=1}^{\lfloor \log_2 x \rfloor} x^{1/k} \left(\frac{\log x}{k} - 1 \right) + O(x^c \log x)$.

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Remark

$Q(x) = \pi_{\mathbb{S}}(x) + 1$ *so it's sufficient to work with* $\pi_{\mathbb{S}}(x)$

A Necessary Lemma

Lemma

For $x \geq 0$ and assuming RH, we have

$$\psi_{\mathbb{S}}(x) - \vartheta_{\mathbb{S}}(x) = \frac{6}{\pi^2} \sum_{k=2}^{\lfloor \log_2 x \rfloor} x^{1/k} \left(\frac{\log x}{k} - 1 \right) + O(x^{\frac{1}{6}} \log x \log \log x).$$

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Remark

This is not proved with any assumptions of asymptotics, only from the definitions of $\psi_{\mathbb{S}}(x)$ and $\vartheta_{\mathbb{S}}(x)$ and known bounds on $Q(x)$.

Proof Sketch of Main Theorem (1/3)

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Proof Sketch.

- We show (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3)

Proof Sketch (2/3)

Proof Sketch.

- (1) $\Rightarrow$ (2): Use partial summation on

$$\vartheta_{\mathbb{S}}(x) = \sum_{\substack{n \in \mathbb{S} \\ n \leq x}} \log n = \sum_{n \leq x} |\mu(n)| \log n$$

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- (2) $\Rightarrow$ (1): Use partial summation on

$$\pi_{\mathbb{S}}(x) = \sum_{2 \le n \le x} |\mu(n)| = \sum_{2 \le n \le x} |\mu(n)| \log n \cdot \frac{1}{\log n}$$

Proof Sketch (3/3)

Proof Sketch.

- (2) $\Rightarrow$ (3): Use previous lemma on

$$\psi_{\mathbb{S}}(x) = \vartheta_{\mathbb{S}}(x) + \left[\psi_{\mathbb{S}}(x) - \vartheta_{\mathbb{S}}(x)\right]$$

Proof Sketch (3/3)

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The Squarefree Partition Zeta Function

Lemma

The zeta function $\zeta_{\mathcal{P}_{\mathbb{S}}}(s)$ can be expressed in terms of the classical zeta function as

$$\zeta_{\mathcal{P}_{\mathbb{S}}}(s) = \exp\left\{\sum_{k\geq 1} \frac{1}{k}\left(\frac{\zeta(ks)}{\zeta(2ks)} - 1\right)\right\},$$

for $\mathrm{Re}(s) > 1$.

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Corollary

The logarithmic derivative of $\zeta_{\mathcal{P}_{\mathbb{S}}}(s)$ is given by

$$\frac{\zeta'_{\mathcal{P}_{\mathbb{S}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{S}}}(s)} = \frac{d}{ds}\sum_{k\geq 1} \frac{1}{k}\frac{\zeta(ks)}{\zeta(2ks)}.$$

Conjectured Exact Formula

Conjecture

The function $\psi_{\mathbb{S}}(x)$ has "exact" form

$$\psi_{\mathbb{S}}(x) = \frac{6}{\pi^2} \sum_{k=1}^{\log_2 x} x^{\frac{1}{k}} \left(\frac{\log x}{k} - 1 \right) + O(x^{\frac{1}{4}+\varepsilon} \log x).$$

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A Possible Starting Point

From PT Perron's Formula and our log derivative result

$$\widetilde{\psi}_{\mathbb{S}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{d}{ds} \sum_{k \geq 1} \frac{1}{k} \frac{\zeta(ks)}{\zeta(2ks)} \right) \frac{x^s}{s} ds.$$

Computational Verification

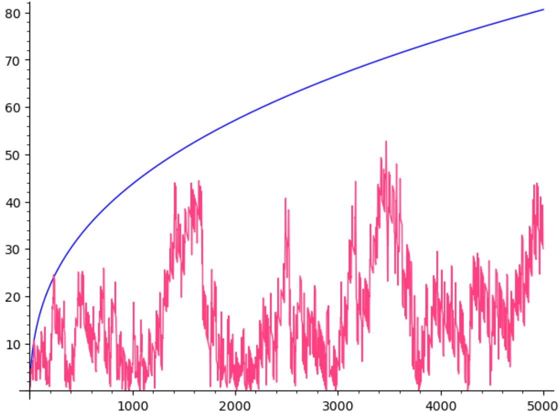

Figure: Absolute Error between approximation $\frac{6}{\pi^2}\sum_{k=1}^{\lfloor \log_2 x \rfloor} x^{\frac{1}{k}} \left(\frac{\log x}{k} - 1\right)$ and $\psi_{\mathbb{S}}(x)$ (red) against conjectured error bound of $1.13x^{\frac{1}{4}} \log x$ (blue)

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Partitions Into Distinct Parts

Definition

We call $\lambda \in \mathcal{P}$ a **partition into distinct parts** if $\lambda_i \neq \lambda_j$ for any $i \neq j$ and $\lambda_i, \lambda_j \in \lambda$, i.e. every part of λ is distinct.

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$$\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s) := \sum_{\lambda \in \mathcal{P}^*_{\mathbb{X}}} n_\lambda^{-s} = \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \frac{|\mu_{\mathcal{P}}(\lambda)|}{n_\lambda^s}$$

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- We have an Euler product $\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s) = \prod_{n \in \mathbb{X}} \left(1 + n^{-s}\right)$

Analogs for Distinct Partition Zeta Functions

"Squarefree" Zeta Function	Distinct Part Partition Zeta Function
$\zeta_{\mathrm{sf}}(s) := \sum_{n \geq 1} \frac{\lvert\mu(n)\rvert}{n^s}$	$\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s) := \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \frac{\lvert\mu_{\mathcal{P}}(\lambda)\rvert}{n_\lambda^s}$
$\zeta_{\mathrm{sf}}(s) = \prod_{p \in \mathbb{P}} (1 + p^{-s})$	$\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s) = \prod_{n \in \mathbb{X}} (1 + n^{-s})$
$\zeta_{\mathrm{sf}}(s) = \frac{\zeta(s)}{\zeta(2s)}$	$\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s) = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(2s)}$

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$\zeta_{sf}(s) = \frac{\zeta(s)}{\zeta(2s)}$	$\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s) = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(2s)}$

$$\frac{\zeta_{sf}'(s)}{\zeta_{sf}(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)\Lambda(n)}{n^s} \longleftrightarrow \frac{\zeta_{\mathcal{P}_{\mathbb{X}}^*}'(s)}{\zeta_{\mathcal{P}_{\mathbb{X}}^*}(s)} = \sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \frac{(-1)^{\ell(\lambda)}\Lambda_{\mathcal{P}}(\lambda)}{n_\lambda^s}$$

The Signed Chebyshev Functions

Definition

We define the **signed Chebyshev function** by

$$\Psi(x) := \sum_{n \leq x} (-1)^{\Omega(n)+1} \Lambda(n) = \sum_{n \leq x} -\lambda(n)\Lambda(n),$$

where $\Omega(n)$ denotes the number of prime factors (counting multiplicity) of n and $\lambda(n)$ is the Liouville lambda function.

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where $\Omega(n)$ denotes the number of prime factors (counting multiplicity) of n and $\lambda(n)$ is the Liouville lambda function. We define the **partition-theoretic signed Chebyshev function** by

$$\Psi_{\mathcal{P}}(x) := \sum_{\substack{\lambda \in \mathcal{P}, \\ n_\lambda \leq x}} (-1)^{\ell(\lambda)+1} \Lambda_{\mathcal{P}}(\lambda).$$

An Exact Formula for Ψ

Proposition

Let $\mathbb{X} \subseteq \mathbb{Z}^+$*. If* $\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} (-1)^{\ell(\lambda)+1} \Lambda_{\mathcal{P}}(\lambda) n_\lambda^{-s}$ *converges for* $\mathrm{Re}(s) > \sigma$*, then for* $c > \sigma$ *we have*

$$\widetilde{\Psi}_{\mathbb{X}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'_{\mathcal{P}^*_{\mathbb{X}}}(s)}{\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s)} \right) \frac{x^s}{s} ds = \widetilde{\psi}_{\mathbb{X}}(x) - 2\widetilde{\psi}_{\mathbb{X}}(x^{\frac{1}{2}}).$$

Relating $\Psi(x)$ and $\psi(x)$

Corollary

$$\widetilde{\Psi}(x) = \sum_{n\leq x}{}' (-1)^{\Omega(n)+1}\Lambda(n)$$

$$= \psi_0(x) - 2\psi_0\left(x^{\frac{1}{2}}\right)$$

$$= \left(x - 2x^{\frac{1}{2}}\right) + \sum_{\substack{\zeta(\rho)=0 \\ nontrivial}} \frac{2x^{\frac{\rho}{2}} - x^{\rho}}{\rho} + \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + \log(2\pi)$$

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Remark

The results above imply the surprising fact $\Psi(x) \sim \psi(x) \sim x$.

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- We defined partition-theoretic analogs of the von Mangoldt and Chebyshev functions and showed they satisfy analogous properties to their classical counterparts

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- We investigated a potential method to bound the error on the squarefree counting function using our partition-theoretic analogs of the PNT
- Informed by restricting our previous results to distinct part partitions, we defined the signed Chebyshev functions and showed their relationship with the Chebyshev ψ function

Future Research

There are multiple direct avenues of future research:

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We also have broader topics and potential applications of partition-theoretic methods:

- General analytic continuation of partition zeta functions
- PT exponential sums using an analog of Vaughan's identity

Thank You!