Dirichlet Generating Functions and Number Theory

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The Möbius function

Definition

The **Möbius function** is the number-theoretic function given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 | n \\ (-1)^s & \text{if } n \text{ is the product of s distinct primes} \end{cases}$$

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Example

Some relevant values of μ for a prime number p are

$$\mu(1) = 1, \quad \mu(p) = (-1)^1 = -1, \quad \mu(p^k) = 0 \text{ for } k \geq 2.$$

The Sum of Divisors Function

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Example

For a prime number p

$$\sigma(1) = 1, \quad \sigma(p) = 1 + p, \quad \sigma(p^k) = 1 + p + p^2 + \ldots + p^k.$$

The Euler Totient Function

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Example

For a prime number p

$$\phi(1) = 1, \quad \phi(p) = p - 1, \quad \phi(p^k) = p^k - p^{k-1}.$$

Dirichlet Product

Definition

For two number-theoretic functions f, g, we define the **Dirichlet product** of f and g by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) = \sum_{d|n} f(n/d)g(d).$$

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Remark

We have the following relevant Dirichlet products

$$(\mu * \sigma)(n) = n, \qquad (1 * \phi)(n) = n.$$

The Riemann Zeta Function

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For $s \in \mathbb{C}$ with $\sigma > 1$, we define the Riemann zeta function

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Theorem (Euler Product Representation)

We have the following representation of ζ as an infinite product over the prime numbers,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Dirichlet Generating Functions

Definition

A **Dirichlet generating function** (DGF) is a series of the form

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where $s \in \mathbb{C}$ and $f(n)$ is an arithmetic function $f : \mathbb{N} \to \mathbb{C}$.

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Example

The zeta function is the DGF over $f(n) = 1$ for all $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Dirichlet Generating Functions

Proposition

The DGF of a multiplicative function, f, can be represented as an Euler product, namely

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right),$$

where the product is over all prime numbers.

Product of DGFs

Lemma

Let $F(s) = \sum_{n=1}^{\infty} f(n)/n^s$ *and* $G(s) = \sum_{n=1}^{\infty} g(n)/n^s$ *be two Dirichlet series. Then, we have*

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}.$$

DGF for μ

Theorem

The Dirichlet generating function for μ *is*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Proof of μ DGF

Proof.

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \ldots\right)$$

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Proof of μ DGF

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$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \ldots\right)$$
$$= \prod_p \left(1 + \frac{-1}{p^s} + \frac{0}{p^{2s}} + \ldots\right)$$

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Theorem

The Dirichlet generating function for σ is

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Remark

Since we have shown that the DGF for μ is $1/\zeta(s)$, we will prove

$$\frac{1}{\zeta(s)} \cdot \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s-1).$$

Proof of σ DGF

Proof.

$$\frac{1}{\zeta(s)} \cdot \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

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$$= \sum_{n=1}^{\infty} \frac{(\mu * \sigma)(n)}{n^s}$$

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Proof of σ DGF

Proof.

$$\begin{aligned}
\frac{1}{\zeta(s)} \cdot \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \\
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&= \sum_{n=1}^{\infty} \frac{n}{n^s}
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Remark

We will show the equivalent statement

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \zeta(s-1).$$

Proof of ϕ DGF

Proof.

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

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$$=\sum_{n=1}^{\infty}\frac{1*\phi(n)}{n^s}$$

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- Proven analytically using DGFs and new number-theoretic tool of "partition-theoretic analogies"
- Combinatorial interpretation is a corollary of the analytic results

Pentagonal Numbers

Definition

The ***n*th pentagonal number**, denoted g_n, is the number of dots in the outlines of nested regular pentagons with side lengths ranging from 1 to n dots and sharing one vertex.

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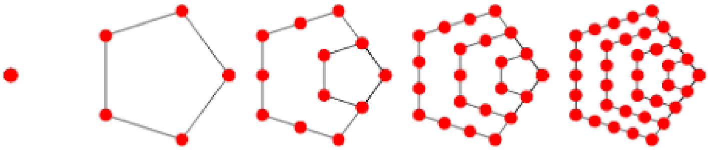
Example

The first few are $g_1 = 1$, $g_2 = 5$, $g_3 = 12$, $g_4 = 22$, $g_5 = 35$.



Pentagonal Number Theorem

Theorem (Euler's Pentagonal Number Theorem)

We have that

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{-\infty}^{+\infty}(-1)^n q^{n(3n+1)/2}$$

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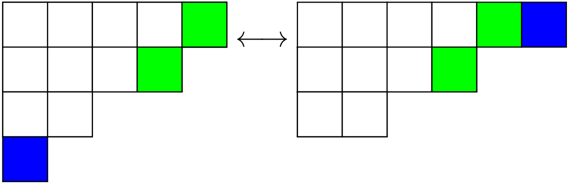

Combinatorial Interpretation of PNT

Notation

Let

- $D^+(n)$ denote the number of partitions of n into an even number of distinct parts,
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Theorem (“Pseudo-bijection”)

$$D^+(n) - D^-(n) = \begin{cases} (-1)^j & \textit{if } n \textit{ is a pentagonal number} \\ 0 & \textit{otherwise.} \end{cases}$$

Defining Our Restricted Partitions

Definition (Ono, Schneider, Wagner)

Let

1. $D^{+}_{even}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part even,
2. $D^{+}_{odd}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part odd,

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3. $D^-_{even}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part even,
4. $D^-_{odd}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part odd.

Statement of Partition Relation

Theorem (Ono, Schneider, Wagner)

For distinct partitions whose smallest part is odd, we have

$$D_{odd}^{+}(n) - D_{odd}^{-}(n) = \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n \text{ is an even square} \\ -1 & \text{if } n \text{ is an odd square.} \end{cases}$$

Statment of Partition Relation Cont.

Theorem (Ono, Schneider, Wagner)

For distinct partitions whose smallest part is even, we have

$$D^{+}_{even}(n) - D^{-}_{even}(n)$$
$$= \begin{cases} -1 & n \text{ is an even square \& not a pentagonal number} \\ 1 & n \text{ is an odd square \& not a pentagonal number} \\ 1 & n \text{ is an even index pentagonal number \& not a square} \\ -1 & n \text{ is an odd index pentagonal number \& not a square} \\ 0 & \text{otherwise.} \end{cases}$$

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$\mu(m)$	$\mu_{\mathcal{P}}(\lambda)$
m^{-s}	$q^{\lvert\lambda\rvert}$

A Look at the Partition-Theoretic DGF

Definition

The **partition-theoretic Möbius function** is defined as

$$\mu_{\mathcal{P}}(\lambda) := \begin{cases} 0 & \text{if } \lambda \text{ has repeated parts} \\ (-1)^{\ell(\lambda)} & \text{otherwise} \end{cases}$$

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Remark

Using that the analogy for n^{-s} is $q^{|\lambda|}$, we have that the DGF for $\mu_{\mathcal{P}}$ is

$$\sum_{\lambda \in \mathcal{P}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = \prod_{n=1}^{\infty} (1 - q^n).$$

Main Theorem

Theorem (Ono, Schneider, Wagner)

The following identities are true

$$\sum_{\substack{\lambda \in \mathcal{P} \\ sm(\lambda) \in S_{1,2}}} -\mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2},$$

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$$\sum_{\substack{\lambda\in\mathcal{P}\\ sm(\lambda)\in S_{2,2}}} -\mu_{\mathcal{P}}(\lambda)q^{|\lambda|} = 1+\sum_{n=1}^{\infty}(-1)^{n}q^{n^2} - \sum_{m=-\infty}^{\infty}(-1)^{m}q^{\frac{m(3m-1)}{2}}.$$

Conclusion

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- They have beautiful relations to both combinatorial and analytic objects.
- They are still being used widely in number theory and combinatorics.

Thank You!