Analytic Continuation of the Riemann Zeta Function

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The Riemann Zeta Function

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For $s \in \mathbb{C}$ with $\sigma > 1$, we define the Riemann zeta function

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Theorem (Euler Product Representation)

We have the following representation of ζ as an infinite product over the prime numbers,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Pretty Pictures?

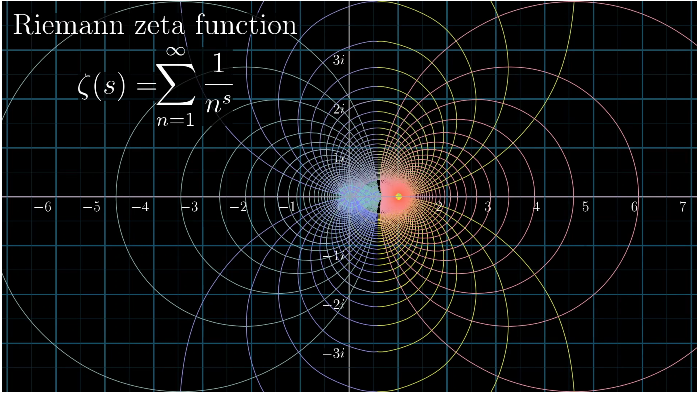

Figure: 3Blue1Brown

Mellin Transforms

Definition

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a continuous function which rapidly decays at ∞ and satisfies $f \in O(t^{-C})$ as $t \to 0$. We define the Mellin transform of f by

$$\mathcal{M}\{f\}(s) = \int_0^\infty f(t)t^{s-1}dt = \int_0^\infty f(t)t^s \frac{dt}{t}.$$

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Remark

$\mathcal{M}\{f\}(s)$ converges absolutely and normally for $\mathrm{Re}(s) > C$.

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Mellin Principle

Let $\phi\colon \mathbb{R}_+ \to \mathbb{C}$ be a continuous function satisfying

$$\phi\left(\frac{1}{t}\right) = \sum_{j=1}^{J} A_j t^{\lambda_j} + t^h \phi(t).$$

Then, $\mathcal{M}\{\phi\}(s)$ has a meromorphic analytic continuation to all of $\mathbb{C}$, with poles at $s = \lambda_1, \ldots, \lambda_J$.

The Gamma Function

Definition

The Gamma function is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,$$

and satisfies $\Gamma(s+1) = s\Gamma(s)$.

Remark

We can note that

$$\mathcal{M}\{e^{-t}\}(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} = \Gamma(s).$$

A Theta Function

Let $\phi(t) = \sum_{n\geq 1} e^{-\pi n^2 t}$. We claim (without proof) that $\phi(t)$ satisfies the functional equation required to apply the Mellin principle. As such, we consider

$$\begin{aligned}\mathcal{M}\{\phi\}(s) &= \int_0^\infty t^{s-1} \cdot \sum_{n=1}^{\infty} e^{-\pi n^2 t} dt \\ &= \sum_{n=1}^{\infty} \int_0^\infty t^{s-1} e^{-\pi n^2 t} dt.\end{aligned}$$

A Theta Function

Now consider the substitution $\pi n^2 t = x$ so $\pi n^2 dt = dx$, then

$$\begin{aligned}
\mathcal{M}\{\phi\}(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{x}{\pi n^2}\right)^{s-1} e^{-\pi x} \pi n^2 dx \\
&= \pi^{-s} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \int_0^{\infty} x^{s-1} e^{-x} dx \\
&= \pi^{-s} \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \Gamma(s) \\
&= \pi^{-s} \Gamma(s) \zeta(2s).
\end{aligned}$$

Thus, $\mathcal{M}\{\phi\}(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ has a meromorphic continuation to $\mathbb{C}$.

Completed Zeta Function

Using the previous results, we can define the analytic continuation for $\zeta(s)$ and the completed zeta function.

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The **completed zeta function** is the entire function given by

$$\xi(s) := \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

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Theorem

The function $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ with a simple pole $s = 1$ and with functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Riemann Hypothesis

Facts about $\zeta(s)$

- Under its analytic continuation, $\zeta(s)$ has zeros at every negative even integer since $\Gamma(s)$ has poles at every integer.

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Conjecture (Riemann)

Apart from the negative evens, the zeros of $\zeta(s)$ satisfy $\operatorname{Re}(s) = \frac{1}{2}$.

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- Using the same process as above, we can analytically continue more general L-functions, which replace the 1 in the numerator of $\zeta(s)$.

Thank You!